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REGIONAL GROUNDWATER MOTION IN RESPONSE TO AN OSCILLATING WATER TABLE

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ABSTRACT

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A case is made for the use of simple models of groundwater flow emphasizing the salient regional characteristics of groundwater basins and for relating them to basin-wide features of groundwater flow. As an example a simple formula for the "natural basin yield" as defined by Freeze is calculated from a model which was introduced by Tóth. The Tóth model is then extended to take seasonal fluctuations of the water table into account; the type of response of the instantaneous head distribution in the basin due to elastic storage in the aquifer may range from severe distortions of Tóth's steady-state distribution to virtually no distortion. The type of response depends on two ratios which can be easily calculated from the basin parameters.

INTRODUCTION

In the management of groundwater and specifically in the protection of this basic resource from contamination, hydrologists rely heavily on models predicting or simulating hydraulic head in the subsurface, enabling them to predict flow paths and transient times of contaminant and thus to design and implement remedial action. More and more reliance is placed on sophisticated numerical models, but in regional settings (e.g., groundwater basins) efficient use can be made of analytic solutions to the differential equation of groundwater flow, since in such a broad context local basin properties, which are usually very inhomogeneous, are of less importance than the average properties of the basin as a whole; therefore, the groundwater manager is often able to, and should, replace a highly complicated system by an average system of simple configuration and homogeneous properties. Models of such an averaged system have distinct advantages: (1) they are less costly and less time-consuming to operate; (2) they are more likely to yield an insight into the physical parameters or combinations of parameters which affect the response of the basin. These advantages are quite obvious if one considers the number of calibration runs that usually must be performed with numerical models before a satisfactory model is obtained, and the tedious work involved in sensitivity analyses of these models.

In the following sections this basic idea of obtaining regional characteristics of a groundwater basin will be explored in more detail, based on a simplified basin representation which was conceived by Tóth (1962) to explain his observations of local features of groundwater motion on the western part of the Canadian Plains. Tóth (1963a) later extended the theory to a "composite" basin consisting of a series of sinusoidal, minor valleys and water divides superimposed on the slope of a larger regional basin. Freeze (1969) extended the same ideas further into multi-layered basins but in order to do so had to make use of numerical techniques for the solution of the differential equation of groundwater flow. Thus, both Tóth and Freeze went from a global to a more specific, precise basin representation, whereas in this paper the emphasis will be on the response of the basin as a whole.

TÓTH's (1962) MODEL OF A DRAINAGE BASIN AND NATURAL BASIN YIELD

It is now more than a decade and a half since Tóth (1962) published the results of his studies of groundwater flow in small drainage basins, thereby introducing some radically new concepts into the description of regional groundwater flow. Flow concepts were until then mainly of an intuitive and qualitative character and rigorous mathematical–physical treatment was limited to problems in well hydrology and drainage. Tóth showed that under certain conditions of geology and topography, which he found in the gently rolling plains of central Alberta, closed systems of groundwater motion must exist, bounded laterally by a water divide and an adjacent valley bottom. Observation further showed that the water table in these valleys generally followed the ground surface and could be approximated by an inclined plane. By stipulating further: (1) that at some depth a subhorizontal formation of relatively low permeability exists; (2) that there is no appreciable flow component in the direction parallel to the valley bottom; and (3) that the actual trapezoidal flow domain may be replaced by a rectangular domain, then a two-dimensional steady-state boundary value problem was defined, permitting an explicit solution for the groundwater head (see also the Notation for symbols used in this paper):

$$H_0 = B/2 - 4B \sum_{m=1}^{\infty} \cos(\beta_m x/L)\cosh(\beta_m y/L)/[\beta_m^2 \cosh(\beta_m D/L)] \qquad (1)$$

where

$H_0(x,y)$	$= \phi/g$, the groundwater head, with reference point ($H_0 = 0$) at the valley bottom	(L)
ϕ	$= gy + \int_{p_0}^{p} (\mathrm{d}p/\rho)$, the fluid potential	(L^2/T^2)
g	= acceleration of gravity	(L/T^2)
B	= elevation difference between divide and valley bottom	(L)

β_m	$= (2m - 1)\pi$	
x	= horizontal coordinate	(L)
L	= horizontal distance between divide and valley bottom	(L)
ρ	= density of the water	(M/L^3)
y	= vertical coordinate	(L)
D	= depth of horizontal impermeable base below valley bottom	(L)

Eq. 1 is taken from Tóth (1962) with only minor changes in notation. Fig. 1, reproduced here from Tóth's (1962) paper, again with only minor changes in notation, shows the actual and theoretical regions of flow, while Fig. 2, also

NOTATION

List of symbols

B	elevation difference between valley bottom and divide	(L)
b	B/L	(dimensionless)
C	amplitude of the cyclic fluctuation of the water table	(L)
c	C/L	(dimensionless)
D	depth to the impermeable boundary	(L)
d	D/L	(dimensionless)
F	$F(x/L, y/D, \omega, 2\pi t/P)$, a function describing the deviation of the head distribution from the steady-state distribution	
f	$f(D/L)$, a function describing the dependence of the natural basin yield on D/L	
g	acceleration of gravity	(L/T^2)
H	$H(x, y, t)$, the nonsteady groundwater head	(L)
H_0	$H_0(x, y)$, the steady-state groundwater head	(L)
h	H/L	(dimensionless)
h_0	H_0/L	(dimensionless)
K	permeability	(L/T)
L	length of the basin along the x-axis	(L)
P	period of the cyclic fluctuation of the water table	(T)
Q	total natural basin yield per unit of time per unit width parallel to the valley bottom	(L^2/T)
S_s	specific storage	(1/L)
s	Laplace transform variable	
t	time	(T)
t'	dimensionless time $Kt/(S_s L^2)$	
U	$h - h_0 = (H - H_0)/L$	(dimensionless)
u	$u(s) = L\{U(t)\}$, Laplace transform of U, $\int_0^\infty \exp(-st)U(t)\,dt$	
x	horizontal coordinate	(L)
x'	x/L	(dimensionless)
y	vertical coordinate	(L)
y'	y/L	(dimensionless)
α_n	$\frac{1}{2}(2n - 1)\pi/d$, where n is a summation index	
β_m	$(2m - 1)\pi$, where m is a summation index	
γ_{nm}	$\alpha_n^2 + \beta_m^2$	
ϕ	fluid potential	(L^2/T^2)
ω	$2\pi S_s L^2/PK$	(dimensionless)
ρ	density of water	(M/L^3)

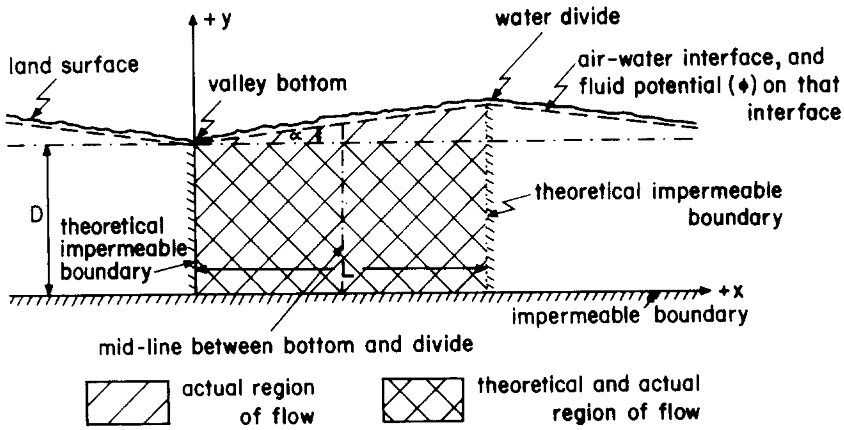

Fig. 1. Cross-section of a valley, showing real and theoretical boundaries and flow regions, from Tóth (1962).

reproduced from Tóth (1962), shows potential distributions and flow lines in two such systems. Some of the important conclusions which Tóth derived from this model are quoted here:

> " . . . the potential distribution is symmetrical with respect to the mid-line between the valley bottom and the water divide. A flow line that originates at the air—water interface at a specific distance uphill from the mid-line will end at the same distance downhill from the mid-line. Thus the discharge is distributed over the area between the mid-line and the valley bottom . . . "

From the viewpoint of groundwater management an important general characteristic is the natural basin yield, since it relates directly to the basin safe yield. Freeze (1969) defined natural basin yield as:

> "The quantity of flow through an undeveloped basin under natural conditions"

and gives an excellent discussion of the relationship between natural and safe yield, and also of the influence of periodic fluctuations of the water table on the natural basin yield to which this paper will turn in the next section.

Basin yield can easily be formulated for the Tóth model from eq. 1 as the integral of the flux across the equipotential line which forms the mid-line (Fig. 2):

$$Q = K\int_0^D \left(\frac{\partial H_0}{\partial x}\right)_{x=L/2} dy$$

where

Q = the total natural basin yield per unit width parallel to the valley bottom per unit of time (L^2/T)

K = the average basin permeability (L/T)

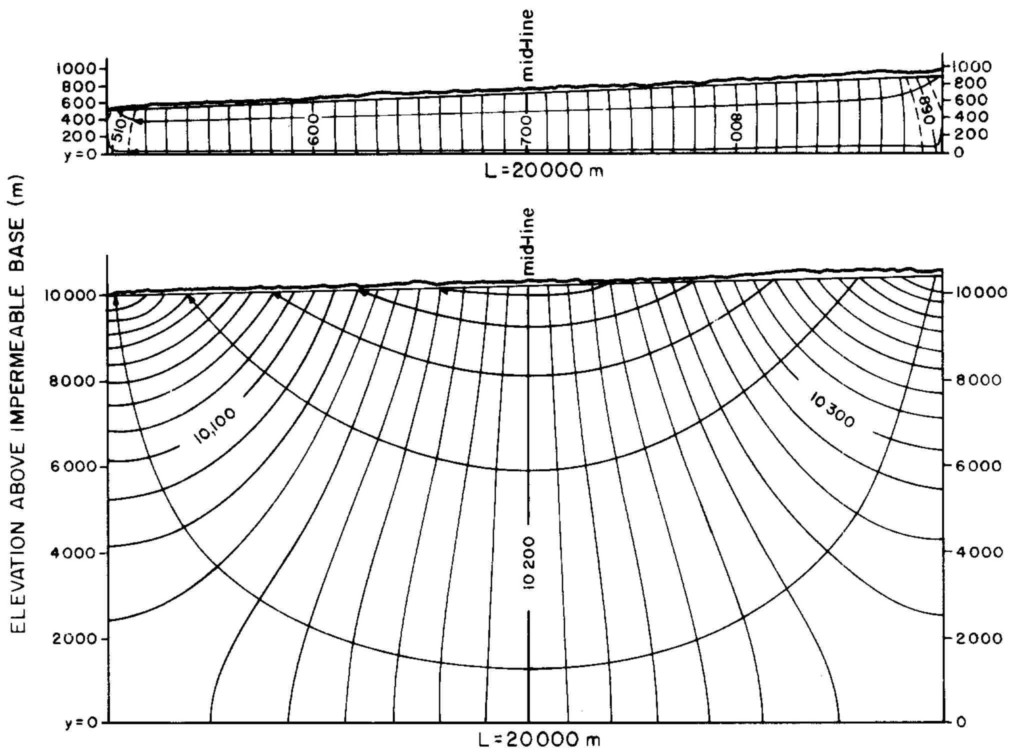

Fig. 2. Two-dimensional theoretical potential distribution and flow patterns for different depths to the horizontal impermeable boundary, from Tóth (1962).

Carrying out the differentiation and integration we obtain:

$$Q = 4BK \sum_{m=1}^{\infty} (-1)^{m-1} \tanh(\beta_m D/L)/\beta_m^2 = BKf(D/L)$$

Thus natural basin yield turns out to be the product of the basin constants B and K and a simple function of the ratio D/L. The function f is shown in Fig. 3. For $D/L \to \infty$, the series approaches to:

$$\lim_{D/L \to \infty} f(D/L) = (4/\pi^2)[1 - (\tfrac{1}{3})^2 + (\tfrac{1}{5})^2 + \ldots] = 0.3712$$

whereas for $D/L \to 0$:

$$\lim_{D/L \to 0} f(D/L) = D/L$$

The latter limit is in agreement with Tóth's (1962) conclusion that for elongated basin shapes all the equipotential lines except those near the valley bottom and divide become nearly vertical and hence the basin yield can be calculated by a simple application of Darcy's equation:

$$Q = KD\,B/L$$

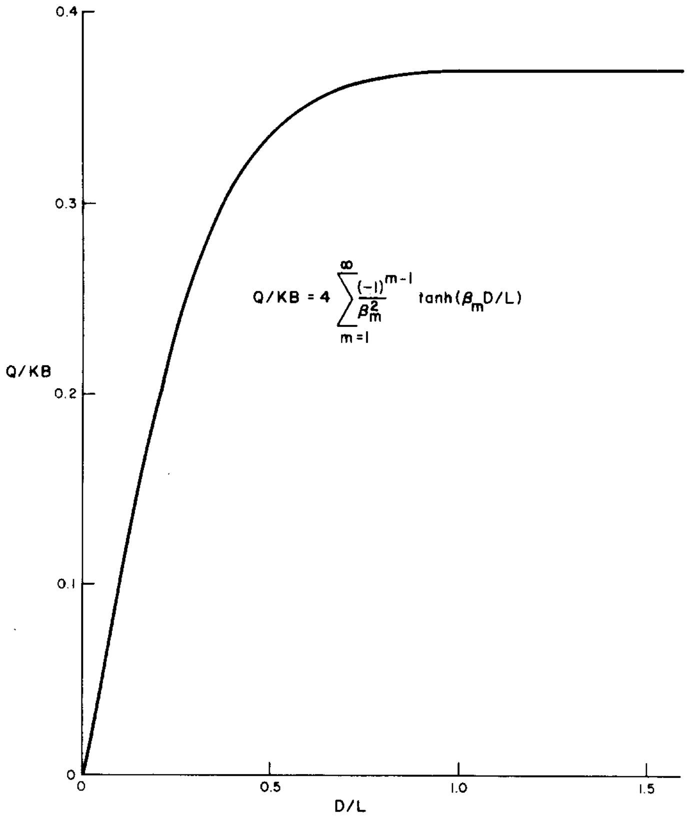

Fig. 3. Dimensionless natural basin yield $Q/(KB)$ as function of the slope of the water table.

THE EFFECT OF A FLUCTUATING WATER TABLE ON BASIN WIDE GROUND-WATER MOVEMENT

Tóth (1962) recognized that the assumption of a steady-state water table is not true, but that instead the water table exhibits a hinge-like seasonal fluctuation, the hinge point being located halfway between divide and valley bottom. In his reply to the objections of Davis (1963), Tóth (1963) stated:

> " . . . the theory gives the long term average of the potential distribution. The theory does not yield quantitatively-transient configurations of the flow pattern . . . "

He justifies the steady-state flow pattern on the ground that the fluctuations of slope of the water table are usually small compared to the average slope and that the mean position of the water table can therefore be taken as the upper boundary of the flow region.

It is interesting, however, to investigate the effect of this type of water table fluctuation a little more closely and to see if gross basin characteristics can give us an indication of how well the steady-state flow pattern describes groundwater motion under actual nonsteady-state conditions. To this end, Tóth's model was extended to include the condition that the water table undergoes cyclic periodic fluctuations with a hinge point at the mid-line (Fig. 4):

$$H(y{=}D) \;=\; B/2 + [(C/L)\sin(2\pi t/P) + B/L](x - L/2)$$

where

x, y, B, C and D have been defined in the steady-state model
H = the nonsteady-state head $H(x,y,t)$ (L)
C = the amplitude of the fluctuation at $x = 0$ and $x = L$ (L)
P = the period of the fluctuation (T)

The method of solution and the complete equation for $H(x,y,t)$ is given in Appendix A. The equation may be written as:

$$H(x,y,t) \;=\; B/2 - 4[C\sin(2\pi t/P) + B] \sum_{m=1}^{\infty} \cos(\beta_m x/L)\cosh(\beta_m y/L)/ [\beta_m^2 \cosh(\beta_m D/L] - (8\omega C/D)F \quad (2)$$

where

ω = $2\pi S_s L^2/PK$ (dimensionless)
S_s = the average specific storage of the basin (1/L)
K = the average permeability of the basin (L/T)
F = a function of x/L, y/D, ω and $2\pi t/P$ given in Appendix A

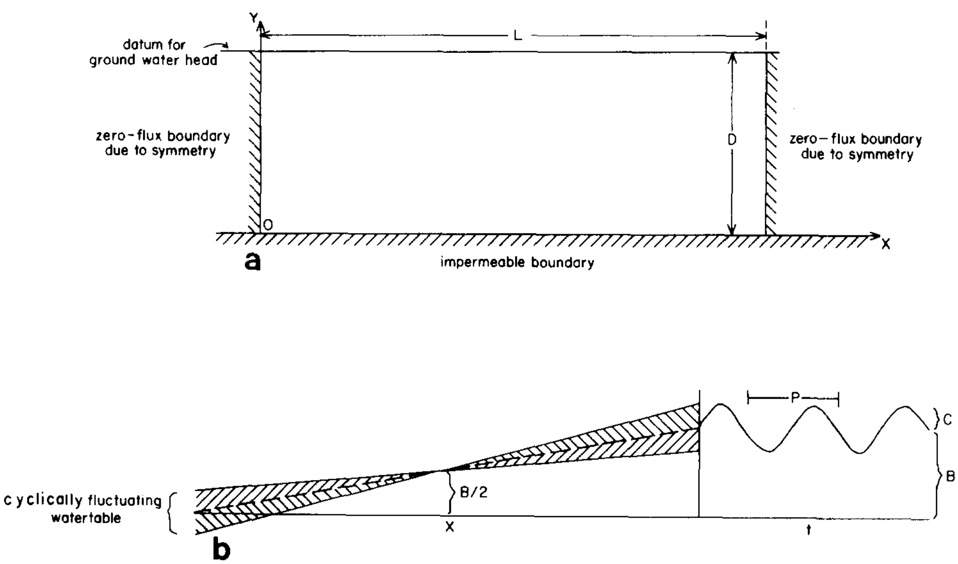

Fig. 4. Theoretical model of a groundwater basin with cyclically fluctuating water table.

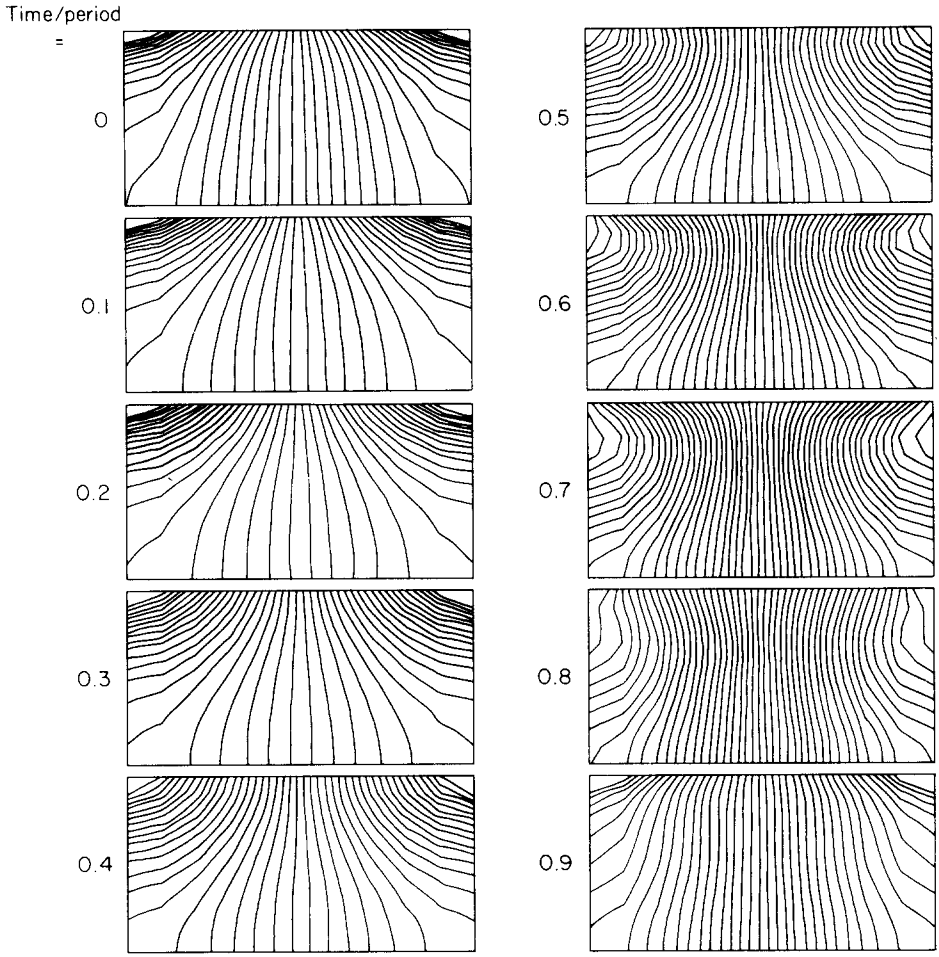

Fig. 5. Groundwater head in a basin throughout a cycle of the water table fluctuation; hinge at the mid-point, $B = 10$, $C = 5$, $D = 5000$, $L = 10{,}000$, $\omega = 120$.

The first two terms of eq. 2 are immediately recognised as the steady-state solution, given the slope of the water table at time t; the third term therefore represents the deviation from the Tóth model. The relative importance of this deviation is thus not only a function of C/B but also of the dimensionless factor ω; this factor can be regarded as the ratio of $S_s L^2/K$ over P, both factors having the dimension of time. The factor $S_s L^2/K$ occurs frequently in groundwater hydrology and can be regarded as the time of propagation of a head disturbance. The ratio $\omega = 2\pi S_s L^2/KP$ is thus the time it takes for the whole basin to respond to the fluctuation of the water table as compared to the period of the fluctuation itself. If ω is small, the basin as a whole will respond quickly and assume a head distribution corresponding closely to the steady-state model at each instant. However, if ω is large, the head change

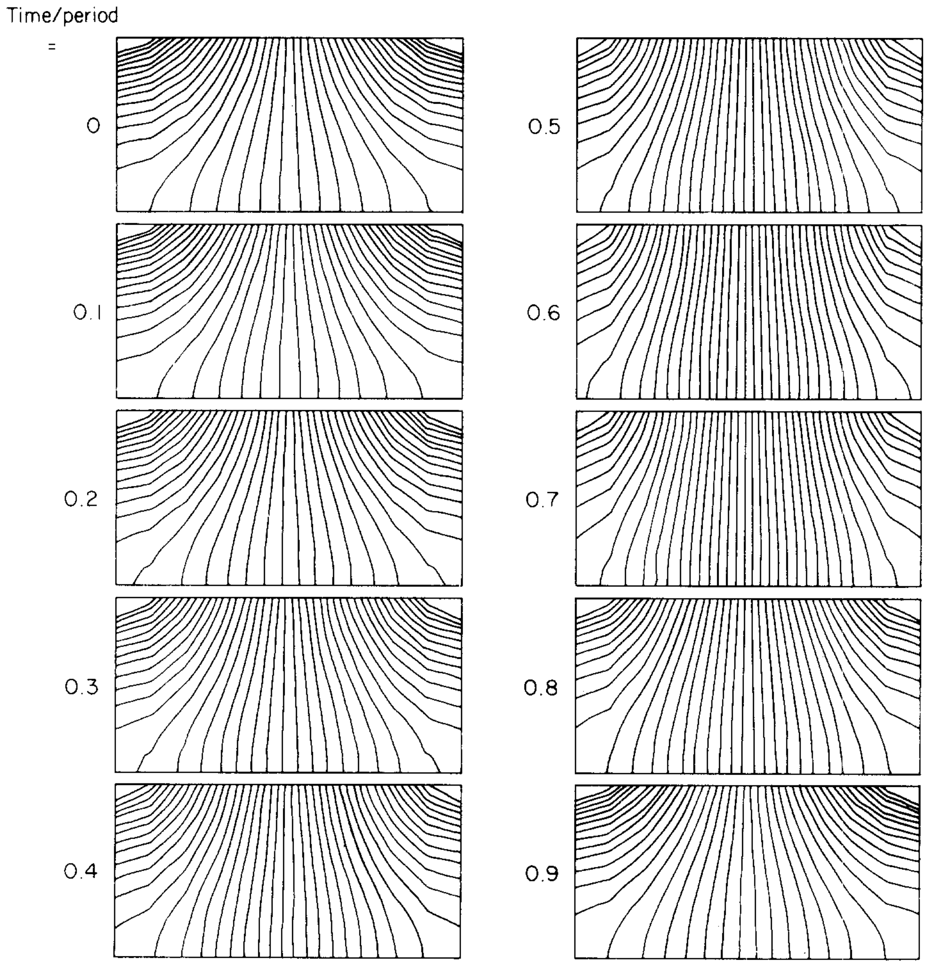

Fig. 6. Groundwater head in a basin throughout a cycle of the water table fluctuation; hinge at the mid-point, $B = 10$, $C = 5$, $D = 5{,}000$, $L = 10{,}000$, $\omega = 12$.

will lag the water table fluctuations by various amounts for various parts of the basin and the resultant instantaneous head distribution will be severely distorted when compared to the steady-state distribution. This phenomenon is clearly illustrated in Figs. 5 and 6 which give the head distributions at ten successive instants of time during a complete cycle for a high and a low value of ω, respectively.

Because of the symmetry of the fluctuation of the water table, eq. 2 is symmetric with respect to the mid-line for all t; an asymmetric head distribution with more severe distortion of the lines of equal head is generated if the hinge point is assumed at the valley bottom as shown in Fig. 7, where the same constants are used as in Fig. 5.

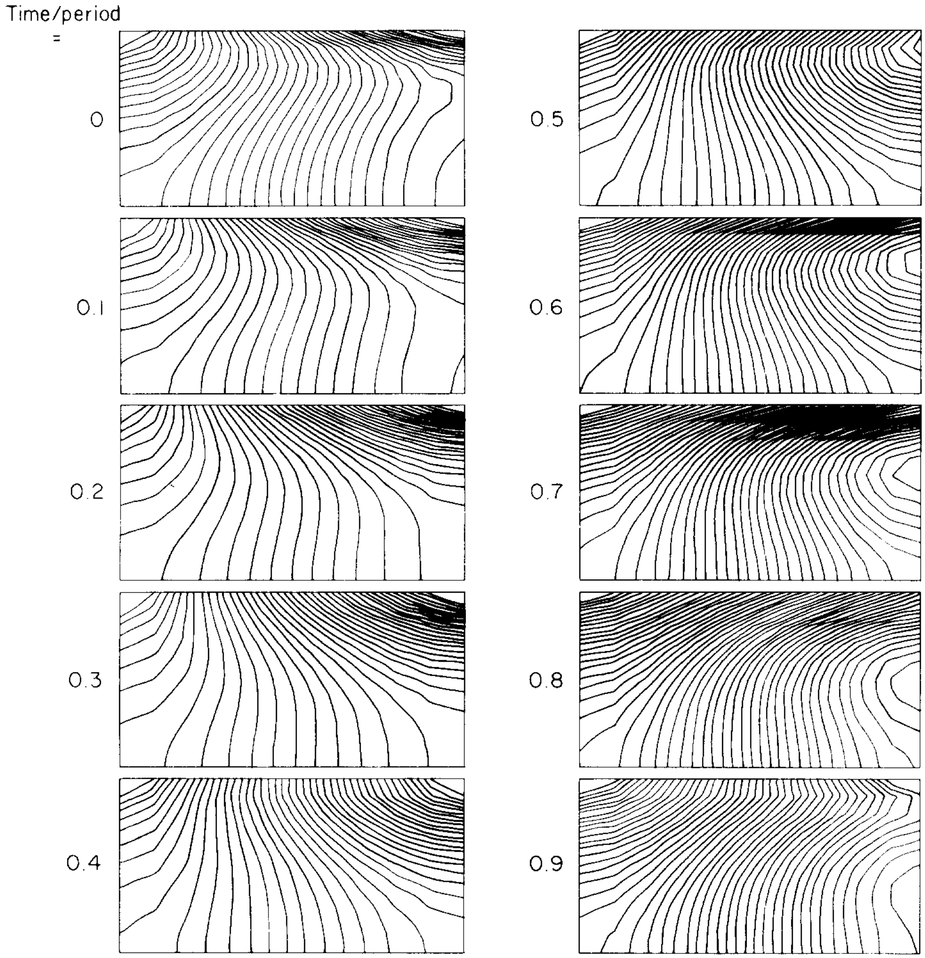

Fig. 7. Groundwater head in a basin throughout a cycle of the water table fluctuation; hinge at the valley bottom, $B = 10$, $C = 5$, $D = 5000$, $L = 10{,}000$, $\omega = 120$.

CONCLUSIONS

Although sophisticated numerical models have proven to be of great assistance in the management of groundwater resources, the study of the behaviour of basins or other groundwater entities by means of simplified models, in which the salient features are emphasized, should not be neglected. Obviously there are risks involved in the latter course: the risks of oversimplification and of missing less salient features which nevertheless may have a large influence on the response of the basin. The assumptions in the extended Tóth model as presented here are subject to criticism; nevertheless the insight gained into the nature of the parameters controlling the amount of the deviation from the steady-state model is applicable regardless of, e.g., the exact shape and fluctuation of the water table (see Appendix B for further discussion).

The constants used in Figs. 5—7 are probably exaggerated, for example, in the area of Central Alberta on which Tóth's (1962) paper is based, approximate values of the parameters are:

$L \simeq 10{,}000\,\text{m}$; $D \simeq 300\,\text{m}$; $B \simeq 200\,\text{m}$; $C \simeq 5\,\text{m}$;
$K \simeq 0.1\,\text{m/day}$; $S_s \simeq 10^{-5}\,\text{m}^{-1}$; $P \simeq 365$ days

from which:

$C/B \simeq 0.025, \quad \omega \simeq 30$

Thus, in this case, although the distortion coefficient ω is high the low value of the ratio C/B indicates that the effect on the head distribution is rather small.

APPENDIX A — SOLUTION OF THE BOUNDARY VALUE PROBLEM FOR GROUND-WATER HEAD IN A BASIN WITH A PERIODICALLY CHANGING WATER TABLE

The boundary value problem

The mathematical model of the flow region is similar to Tóth's (1967) model, with the exception that the slope of the water table is no longer constant:

$$H_0(y{=}D) = Bx/L \tag{A-1}$$

but varies cyclically with time (Fig. 4), the amplitude of the cyclic variation being C at $x = 0$ and $x = L$. Only at the mid-point, $x = L/2$, is the elevation of the water table constant:

$$H(y{=}D) = B/2 + [(C/L)\sin(2\pi t/P) + B/L](x - L/2) \tag{A-2}$$

here

H_0 = the head in the steady-state case
H = the head in the time-variant case
D = the depth to the impermeable boundary
B = the time-averaged height of the water table above the datum plane $y = D$, at the groundwater divide
L = the horizontal distance between the valley bottom and the adjacent divide
t = time
P = the period of the fluctuation of the water table
x, y = space coordinates in the horizontal and vertical direction, respectively

In Fig. 1 the slope of the water table is exaggerated, the real angle being in the order of 1°. As in Tóth's (1962) model the trapezoidal region is replaced by a rectangular region for convenience in the solution of the boundary value problem.

Since at $t = 0$ the head at the water table reduces to:

$$H(y{=}D, t{=}0) = Bx/L = H_0(y{=}D)$$

a suitable initial condition to the boundary-value problem is:

$$H(x, y, 0) = H_0(x, y)$$

where, according to Tóth (1962):

$$H_0(x,y) = B/2 - 4B \sum_{m=1}^{\infty} \cos(\beta_m x/L) \cosh(\beta_m y/L)/[\beta_m^2 \cosh(\beta_m D/L)] \quad \text{(A-2)}$$

with

$$\beta_m = (2m-1)\pi$$

The solution based on this initial condition will, however, contain an unwanted transient which will vanish for $t \to \infty$, leaving only the solution for continuous periodic motion of the water table.

The complete boundary and initial value problem can now be stated as follows: to find the function $H(x, y, t)$ which satisfies the differential equation:

$$\partial^2 H/\partial x^2 + \partial^2 H/\partial y^2 = (S_s/K)\partial H/\partial t \quad \text{(A-3)}$$

subject to the boundary conditions:

$$\partial H/\partial x = 0 \quad \text{at} \quad x = 0 \quad \text{and} \quad x = L \quad \text{(A-4)}$$

$$\partial H/\partial y = 0 \quad \text{at} \quad y = 0 \quad \text{(A-5)}$$

$$H(y{=}D) = B/2 + [(C/L)\sin(2\pi t/P) + B/L](x - L/2) \quad \text{(A-6)}$$

and the initial condition:

$$H(x, y, 0) = H_0(x, y) \quad \text{(A-7)}$$

where H_0 is given by eq. A-2.

Definition of dimensionless variables and auxiliary function U

On introduction of the dimensionless variables:

$$h = H/L, \quad h_0 = H_0/L, \quad x' = x/L$$
$$y' = y/L, \quad b = B/L, \quad c = c/L$$
$$d = D/L, \quad t' = Kt/(S_s L^2), \quad \omega = 2\pi S_s L^2/PK$$

eqs. A-3—A-7 become:

$$\partial^2 h/\partial x'^2 + \partial^2 h/\partial y'^2 = \partial h/\partial t' \quad \text{(A-8)}$$

with the boundary conditions:

$$\partial h/\partial x' = 0 \quad \text{at} \quad x' = 0 \quad \text{and} \quad x' = 1 \quad \text{(A-9)}$$

$$\partial h/\partial y' = 0 \quad \text{at} \quad y' = 0 \tag{A-10}$$

$$h(y'{=}d) = [c\sin(\omega t') + b]x' - c\sin(\omega t')/2 = c(x' - \tfrac{1}{2})\sin(\omega t') + bx' \tag{A-11}$$

and the initial condition:

$$h(x',y',0) = h_0(x',y') = b/2 - 4b\sum_{m=1}^{\infty}\cos(\beta_m x')\cosh(\beta_m y')/[\beta_m^2\cosh(\beta_m d)] \tag{A-12}$$

At this point it is advantageous to introduce the auxiliary function:

$$U = h - h_0 \tag{A-13}$$

thereby reducing the problem to:

Differential equation: $$\partial^2 U/\partial x'^2 + \partial^2 U/\partial y'^2 = \partial U/\partial t' \tag{A-14}$$

with the boundary conditions:

$$\partial U/\partial x' = 0 \quad \text{for} \quad x' = 0 \quad \text{and} \quad x' = 1 \tag{A-15}$$

$$\partial U/\partial y' = 0 \quad \text{for} \quad y' = 0 \tag{A-16}$$

$$U = c\sin(\omega t')(x' - \tfrac{1}{2}) \quad \text{for} \quad y' = d \tag{A-17}$$

and

initial condition: $$U = 0 \quad \text{for} \quad t' = 0 \tag{A-18}$$

Solution

On taking Laplace transforms eqs. A-14—A-16 become:

$$\partial^2 u/\partial x'^2 + \partial^2 u/\partial y'^2 = su \tag{A-19}$$

$$\partial u/\partial x' = 0 \quad \text{for} \quad x' = 0 \quad \text{and} \quad x' = 1 \tag{A-20}$$

$$\partial u/\partial y' = 0 \quad \text{for} \quad y' = 0 \tag{A-21}$$

$$u = c(x' - \tfrac{1}{2})L\{\sin\omega t'\} = c\omega(x' - \tfrac{1}{2})L\{\cos(\omega t')\}/s \quad \text{for} \quad y' = d \tag{A-22}$$

Here $u = u(s)$ = the Laplace transform of $U = L\{U\} = \int_0^\infty \exp(-st')U(t')\mathrm{d}t'$ and s = the transform variable.

The general solution of eq. A-19, subject to eqs. A-20 and A-21 is:

$$u = \sum B_m \cos(m\pi x')\cosh[y'(s + m^2\pi^2)^{1/2}]$$

where the coefficients B_m can be evaluated as:

$$B_0 = 0 \quad \text{and} \quad B_m = -4c\omega L(\sin\omega t')/[\beta_m^2\cosh\{d(s + \beta_m^2)^{1/2}\}]$$

with

$\beta_m = (2m-1)\pi$

Thus:

$$u = -4c\omega \sum_{m=1}^{\infty} [\cos(\beta_m x')/\beta_m^2]\, L\{\cos \omega t'\}(1/s)\cosh[y'(s+\beta_m^2)^{1/2}]/$$
$$\cosh[d(s+\beta_m^2)^{1/2}] \qquad \text{(A-23)}$$

The inverse transform of eq. A-23:

$L^{-1}\{u\} = U$

can be obtained from the known inverse (Spiegel, 1965):

$$L^{-1}\{\cosh(y's^{1/2})/\cosh(ds^{1/2})\} = (2/d)\sum_{n=1}^{\infty}(-1)^{n-1}\alpha_n \cos(\alpha_n y')\exp(-\alpha_n^2 t')$$

where

$\alpha_n = \frac{1}{2}(2n-1)\pi/d$

and general properties of the Laplace transform:

$$U = -4c\sin(\omega t')\sum_{m=1}^{\infty}\cos(\beta_m x')\cosh(\beta_m y')/[\beta_m^2\cosh(\beta_m d] -$$
$$(8\omega c/d)\sum_{m=1}^{\infty}\cos(\beta_m x')/(\beta_m^2)\sum_{n=1}^{\infty}(-1)^n\alpha_n\cos(\alpha_n y')[\gamma_{nm}\cos(\omega t') +$$
$$\omega\sin(\omega t') - \gamma_{nm}\exp(-\gamma_{nm}t')]/[\gamma_{nm}(\gamma_{nm}^2 + \omega^2)] \qquad \text{(A-24)}$$

where

$\gamma_{nm} = \alpha_n^2 + \beta_m^2$

Then, after removal of the transient term in eq. A-24 and replacing U by $h - h_0$:

$$h = U + h_0 = b/2 - 4[c\sin(\omega t') + b]\sum_{m=1}^{\infty}\cos(\beta_m x')\cosh(\beta_m y')/$$
$$[\beta_m^2\cosh(\beta_m d)] - (8\omega c/d)\sum_{m=1}^{\infty}\cos(\beta_m x')/(\beta_m^2)$$
$$\sum_{n=1}^{\infty}(-1)^n\alpha_n\cos(\alpha_n y')[\gamma_{nm}\cos(\omega t') + \omega\sin(\omega t')]/[\gamma_{nm}(\gamma_{nm}^2 + \omega^2)] \qquad \text{(A-25)}$$

The head distribution (25) is symmetric with respect to the vertical through $x = L/2$ since for $x' = \frac{1}{2}$:

$\cos(\beta_m x') = \cos(2m-1)\pi/2 = 0 \qquad$ for all m

and $h(x' = \frac{1}{2}) = b/2$ is constant, and $\cos[\beta_m(1-x')] = \cos(\beta_m x')$.

APPENDIX B — BASIC ASSUMPTIONS OF THE TÓTH MODEL AND ITS EXTENSION

In a discussion of the first draft of this paper with Prof. van der Molen, Editor of the *Journal of Hydrology*, it was brought to the attention of the author that certain assumptions underlying the conceptual models are confusing and need further discussion and clarification; this appendix was prepared for the second draft of the manuscript in an attempt to correct some of the misunderstanding which may result from the concise manner in which the problem is stated in the main text.

Elasticity and compression storage, S_s, *vs. basin storage.* In water balance studies of complete basins the storage term is represented virtually in toto by the drainage and recharge of the aquifer at the moving water table. This storage by far outweighs any possible variations in storage which may be attributed to the compressibility of the pore water and aquifer frame; elastic storage, however, is the essential parameter determining the response of groundwater potential and flow velocities in the subsurface to local perturbations. Pumping wells are a prime example of such perturbations, and it was in this context that the concept of elastic storage was developed. Another disturbance of the potential is caused by precipitation and infiltration to the water table and evapotranspiration and lowering of the water table; it is this type of disturbance which is the subject of the present paper and since the response of the potential in the basin is under investigation, the parameter of interest is the elastic storage, even though the cause of the perturbations are associated with water table fluctuations.

Definition of the domain of the boundary value problem. Problems of groundwater flow involving the water table have traditionally been the domain of civil and agricultural engineering, e.g., flow through earth dams and flow to drains; they also have traditionally dealt with the static case only, on the assumption that inflow equals outflow and total groundwater in storage does not vary. The logical domain of groundwater regime for these problems is a domain which is bounded above by the water table itself, and the main difficulty in this type of problem is the fact that the position of the water table boundary is not a priori known. Indeed, to find the position of the water table has often been the main object of the exercise.

The problems addressed by Tóth are of a slightly different type. Tóth and other hydrologists working in Alberta had *observed* a number of peculiarities of the groundwater potential in drainage basins on the western prairies which could not be explained with the concepts of classical geohydrology. They also observed that the water table across these basins, or rather the potential at points just below the water table, was fairly closely described by a linear relation, or as Tóth expressed it, the water table was found to be a subdued replica of the topography.

With this observation and the assumption of no-flow boundaries along the other three boundaries of Fig. 1, Tóth had a complete boundary value problem which could be solved mathematically. With this observation not only the position of the upper boundary of the domain was fixed, but also the potential on the boundary; thus, given further the differential equation governing flow in the interior and the assumption of no-flow through the other three boundaries of Fig. 1, the complete boundary value problem was defined and could in principle be solved. Tóth then made one further assumption to make the solution tractable; that replacing the inclined upper boundary by a horizontal boundary would not seriously affect the accuracy of the model given the small angle of slope in his real basins. Alternatively, it could have been stated with equal validity that the potential on a horizontal through the lowest points of the basin would not be significantly different from the potential at the water table; thus it could equally well be assumed that in rough outline the potential on this horizontal was known. Thus the boundary value problem was solved. The same arguments form the basis for the mathematical physical boundary value problem described in the paper and therefore form the connection between the hydrologic problem and the mathematical problem described in this paper.

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